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Reccurence Relation

Recursion → A tech of defining a function, set, seq, Algo in the term of itself.

Recursive formula :- A formula which define any term of a sequence in terms of any number of its previous terms or which express any terms of a sequence as a function of its previous terms is called recursive and the relation is called recursive relation.

→ ~~Also called~~ the Difference Equation.

$a_n = a_{n-1} + 1$

Initial $a_0 = 0$

$n = 1$

I_1 : $a_1 = a_0 + 1$

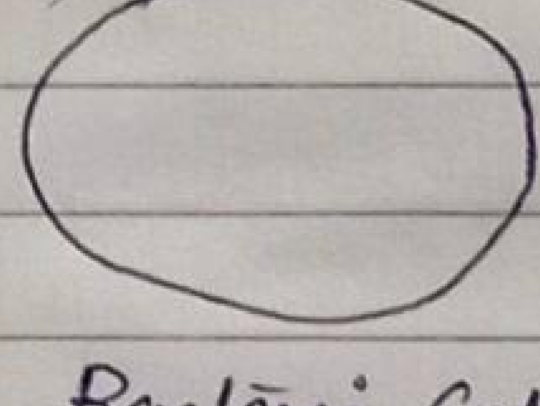
I_2 : $a_2 = a_1 + 1$

⋮ ⋮ ⋮

I_n = $a_n = a_{n-1} + 1$ → Recurrence relation.

Ex :-

"Every hour the no. of bacteria is getting doubled".



Bacteria Colony

which one is the recurrence relation which defines the statement.

(a) $a_n = a_{n-1} + 2$ (b) $a_n = 2a_{n-1}$ ✓ (c) $a_n = a_{n-1} + a_{n-2}$

(d) None.

Recursive Formula: A

There are three steps :-

1. Basic step : we find out the primitive term / initial term

(2) Recursive step: We generate a formula / Rate to find the new term using existing term.

(3) Terminal / final ÷ Verification of formula.

Ex. 2, 9, 16, 23, 30 . . . .

$a_1 = 2$, $a_2 = 9$, $a_3 = 16$, $a_4 = 23$ . . .

$a_2 - a_1 = 7$
$a_3 - a_2 = 7$ ⇒ Generalize
$a_4 - a_3 = 7$

$a_n - a_{n-1} = 7$ $a_1 = 2, n \geq$

$\boxed{a_n = a_{n-1} + 7}$

Recurrence Formula

(step by step)

$a_n = a_{n-1} + 7$; $n \geq 2$, $a_1 = 2$

$n = 2$

$a_2 = a_{2-1} + 7$
$= a_1 + 7$
$= 2 + 7 = 9$

$n = 3$

$a_3 = a_2 + 7$
$= 9 + 7$
$= 16$

General Formula

(Direct)

$a = 2$ $d = 7$

$t_n = a + (n-1)d$

$a_n = 2 + (n-1)7$

$\boxed{a_n = 7n - 5}$ $n \in I$

Ex:- find a_{10}. $n = 10$

$a_{10} = 7(10) - 5$
$= 65$

find $a_{10} = a_9 + 7$

To find a_{10}, we have to find a_9.
So, we cannot find any term directly
So you should move step by step.

In general formula, we directly move to the term we want to find.

Fibonacci Series :- 0, 1, 1, 2, 3 . . .

Primitive / initial

$a_1 = 0$ or $a_0 = 0$
2 $a_2 = 1$ $a_1 = 1$

$a_n = a_{n-1} + a_{n-2} : n \geq 3$
$a_{n+1} = a_n + a_{n-1} : n \geq 2$

Note: The recurrence relation can be written in different form of equation

Ex:-
$a_n = a_{n-1} + 5$
$t_n = t_{n-1} + 5$
$f_n = f_{n-1} = 5$
$f_n = f_{n-1} + 5$

$a_n + a_{n-1} - 2a_{n-2} = 0$
$y_n + y_{n-1} + 2y_{n-2} = 5$
$f_n + f_{n-1} + 2f_{n-2} = 5$
$S_n + S_{n-1} + 2S_{n-2} = 5$

Order of recurrence relation - The Order of recurrence relation or difference equation is defined to be the difference between highest and lowest subscripts of the f(n), or a_n or y_n or f(x) etc.

Ex $y_n + y_{n-1} - 3y_{n-2} + 6y_{n-3} = 5$
Highest $= n$
Lowest $= n-3$

Order $= H - L$
$= n - (n-3)$
$= 3$

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So, this is a Third order reccurence relation.

Degree of recurrence relation or Difference equation :-

The degree of the recurrence relation is defined to be the highest power of $f(n)$ or a_n or s_n or $f(x)$ or y_n.

Ex:- $y^3_{r+3} - 5y_{r+2} + 3y_r - 2y_{r-1} = 0$

Order $= (r+3) - (r-1) = 4$.
So, this is a 4^{th} order recurrence relation.

Degree - 3.

Note: If the degree of recurrence relation is

1) One, it is called linear RR.
2) Two, it is called quad RR.
3) Three, " " " Cubic RR
4) Four, " " " Biquadratic RR.

Homogeneous and Non-homogeneous RR. :- A recurrence relation is called Homogeneous RR if it contain no term that depends upon only on variable 'n' or r Otherwise it is called Non-homogeneous RR.

Linear reccurence relation with constant Coefficient

A recurrence relation of the form $a_1 y_n + a_2 y_{n-1} + \dots = f(n)$ —① where $a_1 \dots a_n$ are cons. called coefficient of R.R. $f(n)$ is a function of n only. is called a linear RR with cons. coeff.

If $f(n) = 0$, It is called homogeneous RR.
$f(n) \neq 0$, It is called non-homo. RR

Ex $a_n = 2a_{n-1} + 5a_{n-2} + 6a_{n-3} + 9$

a) Homogeneous b) Non-homogeneous.

Bcoz $f(n) \neq 0$. { but linear }

Ex. $a_n - 5n a_{n-2} + 6n^2 a_{n-3} = 0$

a) Homogeneous b) Non-homogeneous.

Bcoz $f(n) = 0$. { but not linear }

Ex. $a_n - a_{n-2} + 6a_{n-3} = n^2$

→ Cons. coff and $f(n) \neq 0$
⇒ ~~Non~~ linear non-homogeneous RR

Method to find the solution of linear, homogeneous RR.

① Iteration Method :- In this method the recurrence relation ~~of RR~~ for a_n is used repeatedly to solve for a general solution for a_n in terms of 'n'.

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Q Solve the recurrence Relation $a_n = a_{n-1} + 3 \quad \forall n \geq 1$ where $a_0 = 2$ by iteration method.

Solution :- Here,

$a_0 = 2$ { Primitive }

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_{n-1} \quad a_{n-1} \quad a_n$

$a_n - a_{n-1} = 3 \quad ; n \geq 1$

$a_n = 3 + a_{n-1}$.

{ we want a relationship b/w a_n and a_0.

$$\left\{ \begin{array}{l} a_{n-1} = a_{n-2} + 3 \\ a_{n-2} = a_{n-3} + 3 \\ a_{n-3} = a_{n-4} + 3 \end{array} \right.$$

$a_n = a_{n-1} + 3$

$a_n = [a_{n-2} + 3] + 3$

$a_n = [a_{n-2}] + 2 \times 3$

$a_n = [a_{n-3} + 3] + 2 \times 3$

$= [a_{n-3}] + 3 \times 3$

$a_n = [a_{n-4}] + 4 \times 3$

$\vdots \qquad \vdots$ After iteration

$a_n = [a_{n-i}] + i \times 3$

(A) the term $a_{n-i} = 0 \Rightarrow i = n$

$a_n = a_{n-n} + 3 \times n$

$a_n = a_0 + n \times 3$

$a_n = 2 + 3n \qquad (a_0 = 2 \text{ Given})$

required solⁿ of given RR.

8.4

$a_n = a_{n-1} + 3 \quad n \geq 1$

$a_0 = 2$

$n=1 \quad a_1 = a_0 + 3 = 2+3 = 5$

$n=2 \quad a_2 = a_1 + 3 = 5+3 = 8$

$n=3 \quad a_3 = a_2 + 3 = 8+3 = 11$

$2, 5, 8, 11, 14 \ldots$

$a_n = 3n + 2$

$a_0 = 2$

$a_1 = 3 \times 1 + 2 = 5$

$a_2 = 3 \times 2 + 2 = 8$

A.P

$a_n - a_{n-1} = \underline{k}$

$a_n = k + a_{n-1}$

k = comm. diff. of A.P

$$a_n = \begin{cases} a + (n-1)d \; ; & n \geq 1 \quad \text{i.e., } 1^{st} \text{ term} = a_1 \\ a + nd \; ; & n \geq 0 \quad \text{i.e., } 1^{st} \text{ term} = a_0 \end{cases}$$

$a_n = a_{n-1} + 5$ here $k = 5$ $a_0 = 3$ $a_n = a + nd$
$= 3 + 5n$.

G.P.

$\frac{a_n}{a_{n-1}} = k \Rightarrow a_n = k \cdot a_{n-1}$

$$a_n = \begin{cases} ar^{n-1} \; ; & n \geq 1 \quad 1^{st} \text{ term} = a_1 \\ ar^n \; ; & n \geq 0 \quad 1^{st} \text{ term} = a_0 \end{cases}$$

Ex $a_n = 3a_{n-1} + 5; \quad n \geq 2 \quad a_1 = 2$

$a_n = 3\underbrace{a_{n-1}} + 5$ —— (I) $\qquad a_n = 3a_{n-1} + 5$

$\qquad\qquad\qquad\qquad\qquad\qquad$ & $a_{n-1} = 3a_{n-2} + 5$

$a_n = 3[3a_{n-2} + 5] + 5 \qquad$ || $a_{n-2} = 3a_{n-3} + 5$

$a_n = 3^2 a_{n-2} + (1+3) \times 5$ —— (II)

$a_n = 3^2[3a_{n-3} + 5] + 5(1+3) = 3^3 a_{n-3} + 5(1+3+3^2)$. —— (III)

$a_n = 3^4 a_{n-4} + 5(1+3+3^2+3^3)$

⋮

$a_n = 3^n a_{n-i} + 5(1+3+3^2 \ldots \text{ i terms})$

then a_{n-i} will be a_1 if $n-i = 1$

$i = n-1$

$a_n = 3^{n-1} a_1 + 5[1+3+\ldots (n-1) \text{ terms}]$

$= 3^{n-1} \times 2 + 5[\text{Sum of } (n-1) \text{ terms of G.P} \quad a=1, r=3]$

$S_n = \frac{a(r^n - 1)}{r-1}$

$= 2 \times 3^{n-1} + 5\left[\frac{3^{n-1}-1}{3-1}\right]$ $\qquad S_{n-1} = \frac{a(r^{n-1}-1)}{r-1}$

$= 2 \times 3^{n-1} + \frac{5}{2} 3^{n-1} - \frac{5}{2}$

$a_n = \frac{9}{2} \times 3^{n-1} - \frac{5}{2}, \quad n \geq 1$

Some useful Result

(1) Sum of n terms of an A.P with first term 'a' and common difference 'd'.

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l] \quad l - \text{last term}$$

(2) Sum of n terms of a GP

$a, ar, ar^2 \ldots$

$$S_n = \frac{a(r^n-1)}{r-1}$$

(3) $\Sigma n = 1+2+3+\ldots n = \frac{n(n+1)}{2}$

(4) $\Sigma n^2 = 1^2+2^2+3^2+\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

Linear homogeneous recurrence relation

A RR of the form

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k} = f(n)$$

where $C_0, C_1, \ldots C_k$ are constants called coefficient and $f(n)$ is a function 'n' only. is called linear recurrence relation with cons. coefficient

If $f(n) = 0$, it is called Homo RR
$f(n) \neq 0$ Non-homo RR

Method-2. Method of characteristic equation

Working rule: $a_n = C_0 a_{n-1} + C_1 a_{n-2} + \dots C_k a_{n-k} + \dots$

Step I Transpose all the terms to LHS and write the RR in the form and be sure the equation is complete.

$$a_n - C_0 a_{n-1} - C_1 a_{n-2} - \dots C_k a_{n-k} = 0$$

Step 2: Find the order of RR

here the order is $n-(n-k) = k$

Step III: Write down the characteristic equation

$$x^k - C_0 x^{k-1} - C_1 x^{n-2} - C_3 x^{n-3} \dots C_k x^{n-k} = 0 \quad (1)$$

Replace the term

$[a_n \text{ by } x^n]\ [a_{n-1} \text{ by } x^{k-1}]\ [a_{n-2} \text{ by } x^{k-2}] \dots$

Step IV Solve the characteristic equation and find the roots of CE i.e, the value of x

Step V (a): If all the roots of C.E are real and distinct say, $x = \alpha_1, \alpha_2, \alpha_3 \dots$ then linear independent solution and then the general solution

$$a_n = C_1(\alpha_1)^n + C_2(\alpha_2)^{n-1} + C_3(\alpha_3)^{n-2} + \dots$$

where $C_1, C_2, C_3 \ldots$ are arbitrary constant.

By ~~giving~~ using the given internal condition we can find the values of $C_1, C_2 \ldots C_k$.

(b) If two roots of C.E are equal say α, α then the corresponding general solution is $(C_1 + C_2 n)\alpha^n$.

Note: Corresponding L.I solution will be

$\alpha^n, n\alpha^n$ then General solⁿ

$$a_n = C_1\alpha^n + C_2 n\alpha^n$$
$$= \alpha^n (C_1 + nC_2)$$

(c) In case of complex roots i.e; the roots of the form $\alpha + i\beta$ and $\alpha - i\beta$ or $\alpha \pm i\beta$.

the corresponding General solution will be

$$\boxed{a_n = \rho^n [C_1 \cos\theta n + C_2 \sin\theta n]}$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ $\tan\theta = \frac{\beta}{\alpha}$ $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

$\alpha = \rho\cos\theta$ $\beta = \rho\sin\theta$

Ex Solve the 2nd order recurrence relation

$$a_n = 10a_{n-1} - 9a_{n-2}; \quad n \geq 2 \quad a_0 = 3, a_1 = 11$$

Order $= n - (n-2) = 2$

char. eqⁿ. $x^2 - 10x + 9 = 0$

$(x-9)(x-1) = 0 \Rightarrow x = 1, 9$

$x = 1, 9$ roots are real and distinct.

$a_n = c_1(1)^n + c_2(9)^n = c_1 + c_2 9^n$

$a_n = c_1 + c_2 9^n$ $\qquad a_0 = 3, a_1 = 11$

$a_0 = c_1 + c_2 \times 9^0 = 3$

$\Rightarrow c_1 + c_2 = 3$ — (I)

$a_1 = c_1 + c_2 \times 9$

$11 = c_1 + 9c_2$ — (II)

Solving (I) and (II)

$$\begin{array}{r} c_1 + 9c_2 = 11 \\ -c_1 + c_2 = 3 \\ \hline 8c_2 = 8 \\ \hline c_2 = 1 \end{array}$$

$c_1 = 11 - 9 = 2$

$\therefore$ Soln:- $\boxed{a_n = 2 + 9^n}$ $\quad n \geq 0$

Generating Function : If $S = \{a_0, a_1, \ldots\}$ is a sequence of real or complex numbers, then the power series given by

$$G(S,z) = G(z) = \sum_{i=0}^{\infty} a_i z^i = a_0 + a_1 z + a_2 z^2 + \ldots$$

is called the generating function for the given sequence.

$G(S,z) = G(z) = \sum_{n=0}^{\infty} a_n z^n$ where a_n is the general term of the sequence.

$$G(S,z) = G(z) = \sum_{n=0}^{\infty} a_n z^n$$